eLearning by Dr. M.G.R Educational and Research Institute

TOPIC : TRIGONOMETRY- EXPANSION S- PART-II

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Expansions of $cos^n\theta, sin^n\,\theta$

To remember…..

$$\text{Let } x = \cos\theta + i\,sin\theta, \qquad \frac{1}{x} = cos\theta - i\,sin\theta$$

$$x + \frac{1}{x} = 2\cos\theta, \qquad x - \frac{1}{x} = 2i\,sin\theta$$

$$x^n = cos\,n\theta + i\,sin\,n\theta, \qquad \frac{1}{x^n} = \cos n\theta - i\,sin\,n\theta$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta \;\; \& \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

Write the expansion of $cos^6\,\theta$ in series of cosines of multiples of θ

Solution:

Let $x = \cos\theta + i\,sin\theta$, $\quad \frac{1}{x} = cos\theta - i\,sin\theta$

$$x + \frac{1}{x} = 2\cos\theta, \qquad x - \frac{1}{x} = 2i\,sin\theta$$

$$x^n = cos\,n\theta + i\,sin\,n\theta, \qquad \frac{1}{x^n} = \cos n\theta - i\,sin\,n\theta$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta \;\; \& \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

Consider $(2cos\,\theta)^6 = \left(x + \frac{1}{x}\right)^6$

$$= x^6 + 6c_1 x^5 \frac{1}{x} + 6c_2 x^4 \frac{1}{x^2} + 6c_3 x^3 \frac{1}{x^3} + 6c_4 x^2 \frac{1}{x^4} +$$

$$6c_5 x \frac{1}{x^5} + 6c_6 \frac{1}{x^6}$$

$$= x^6 + 6x^4 + 15x^2 + 20 + 15\frac{1}{x^2} + 6\frac{1}{x^4} + \frac{1}{x^6}$$

$$= \left(x^6 + \frac{1}{x^6}\right) + 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) + 20$$

$$= 2cos6\theta + 6(2cos4\theta) + 15(2cos2\theta) + 20$$

$$\therefore 2^6 cos^6\theta = 2(cos6\theta + 6cos4\theta + 15cos2\theta + 10)$$

$$\therefore cos^6\theta = \frac{1}{32}(cos6\theta + 6cos4\theta + 15cos2\theta + 10)$$

- Write the expansion of $\sin^7 \theta$ in a series of sines of multiples of θ

<u>Solution:</u>

Let $x = \cos\theta + i\sin\theta$, $\quad \frac{1}{x} = \cos\theta - i\sin\theta$

$$x + \frac{1}{x} = 2\cos\theta, \quad x - \frac{1}{x} = 2i\sin\theta$$

$$x^n = \cos n\theta + i\sin n\theta, \quad \frac{1}{x^n} = \cos n\theta - i\sin n\theta$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta \;\&\; \quad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

Consider $(2i\sin\theta)^7 = \left(x - \frac{1}{x}\right)^7$

$$= x^7 - 7c_1 x^6 \frac{1}{x} + 7c_2 x^5 \frac{1}{x^2} - 7c_3 x^4 \frac{1}{x^3} +$$

$$7c_4 x^3 \frac{1}{x^4} - 7c_5 x^2 \frac{1}{x^5} + 7c_6 x \frac{1}{x^6} - 7c_7 \frac{1}{x^7}$$

$$= x^7 - 7x^5 + 21x^3 - 35x + 35\frac{1}{x} - 21\frac{1}{x^3} + 7\frac{1}{x^5} - \frac{1}{x^7}$$

$$= \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right)$$

$$= 2i\ sin7\theta \quad - 7(2i\ sin5\theta) + 21(2i\ sin3\theta) \quad - 35(2i\ sin\theta)$$

$$(i.e)\ \ (2i)^7 sin^7\theta = 2i(\ sin7\theta - 7sin5\theta + 21sin3\theta - 35sin\theta)$$

$$\therefore sin^7\theta = -\frac{1}{64}(\ sin7\theta - 7sin5\theta + 21sin3\theta - 35sin\theta)$$

- Write the expansion of $\sin^4 \theta \cos^5 \theta$ in a series of cosines of multiples of θ

<u>Solution:</u>

$$\text{Let } x = \cos\theta + i\sin\theta, \qquad \frac{1}{x} = \cos\theta - i\sin\theta$$

$$x + \frac{1}{x} = 2\cos\theta, \qquad x - \frac{1}{x} = 2i\sin\theta$$

$$x^n = \cos n\theta + i\sin n\theta, \qquad \frac{1}{x^n} = \cos n\theta - i\sin n\theta$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta \;\; \& \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

Consider, $(2i\sin\theta)^4(2\cos\theta)^5 = \left(x - \frac{1}{x}\right)^4 . \left(x + \frac{1}{x}\right)^5$

$$= \left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)$$

$$= \left[\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)\right]^4 . \left(x + \frac{1}{x}\right)$$

$$= \left(x^2 - \frac{1}{x^2}\right)^4 . \left(x + \frac{1}{x}\right)$$

$$= \left(x^8 + 4c_1(x^2)^3\left(-\frac{1}{x^2}\right) + 4c_2(x^2)^2\left(-\frac{1}{x^2}\right)^2 + \right.$$

$$\left. 4c_3x^2\left(-\frac{1}{x^2}\right)^3 + 4c_4\left(-\frac{1}{x^2}\right)^4\right)\left(x + \frac{1}{x}\right)$$

$$= \left(x^8 - 4\,x^6\frac{1}{x^2} + 6x^4\frac{1}{x^4} - 4x^2\frac{1}{x^6} + \frac{1}{x^8}\right).\left(x + \frac{1}{x}\right)$$

$$= \left(x^8 - 4x^4 + 6 - 4.\frac{1}{x^4} + \frac{1}{x^8}\right).\ \left(x + \frac{1}{x}\right)$$

$$= x^9 - 4x^5 + 6x - 4\frac{1}{x^3} + \frac{1}{x^7} + x^7 - 4x^3 + 6\frac{1}{x} - 4\frac{1}{x^5} + \frac{1}{x^9}$$

$$= \left(x^9 + \frac{1}{x^9}\right) + \left(x^7 + \frac{1}{x^7}\right) - 4\left(x^5 + \frac{1}{x^5}\right) - 4\left(x^3 + \frac{1}{x^3}\right) + 6\left(x + \frac{1}{x}\right)$$

$$= \left(x^9 + \frac{1}{x^9}\right) + \left(x^7 + \frac{1}{x^7}\right) - 4\left(x^5 + \frac{1}{x^5}\right) - 4\left(x^3 + \frac{1}{x^3}\right) + 6\left(x + \frac{1}{x}\right)$$

$$= 2cos9\theta + 2cos7\theta - \mathbf{4(cos5\theta)} - 4(2cos3\theta) + 6(2cos\theta)$$

$(i.e)\ (2i)^4 sin^4\theta . 2^5 cos^5\theta$

$$= 2\left(cos9\theta + cos7\theta - 4(cos5\theta) - 4(cos3\theta) + 6(cos\theta)\right)$$

$$\therefore \sin^4\theta \cos^5\theta = \frac{1}{2^8}\left(cos9\theta + cos7\theta - 4(cos5\theta) - 4(cos3\theta) + 6(cos\theta)\right)$$

Prove that $sin^5\theta\ \ cos^2\theta = \frac{1}{2^6}[sin\ 7\theta - 3sin\ 5\theta + \sin 3\theta + 5sin\theta]$

Solution:

Let $x = \cos\theta + i\ sin\theta$, $\quad \frac{1}{x} = cos\theta - i\ sin\theta$

$$x + \frac{1}{x} = 2\cos\theta, \qquad x - \frac{1}{x} = 2i\ sin\theta$$

$$x^n = cos\ n\theta + i\ sin\ n\theta, \qquad \frac{1}{x^n} = \cos n\theta - i\ sin\ n\theta$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta \ \ \& \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

Consider $(2isin\theta)^5(2cos\theta)^2 \quad = \left(x - \frac{1}{x}\right)^5 . \left(x + \frac{1}{x}\right)^2$

$$= \left(x - \frac{1}{x}\right)^3 \left(x - \frac{1}{x}\right)^2 \left(x + \frac{1}{x}\right)^2$$

$$= \left(x - \frac{1}{x}\right)^3 \left(\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)\right)^2$$

$$= \left(x - \frac{1}{x}\right)^3 \left(x^2 - \frac{1}{x^2}\right)^2 \qquad = \left(x^3 - 3x + 3\frac{1}{x} - \frac{1}{x^3}\right).\left(x^4 - 2 + \frac{1}{x^4}\right)$$

$$= x^7 - 2x^3 + \frac{1}{x} - 3x^5 + 6x - 3\frac{1}{x^3} + 3x^3 - 6\frac{1}{x} + 3\frac{1}{x^5} - x + 2\frac{1}{x^3} - \frac{1}{x^7}$$

$$= x^7 - \frac{1}{x^7} - 3x^5 + 3\frac{1}{x^5} + x^3 - \frac{1}{x^3} + 5x - 5\frac{1}{x}$$

$$= \left(x^7 - \frac{1}{x^7}\right) - 3\left(x^5 - \frac{1}{x^5}\right) + \left(x^3 - \frac{1}{x^3}\right) + 5\left(x - \frac{1}{x}\right)$$

$$= 2i sin\, 7\theta - 3(2i sin\, 5\theta) + 2i \sin 3\theta + 5(2i\, sin\theta)$$

$$(i.e) \qquad (2i)^5 sin^5\theta \;\; 2^2 cos^2\theta = 2i(sin\, 7\theta - 3sin\, 5\theta + \sin 3\theta + 5sin\theta)$$

$$\therefore sin^5\theta \;\; cos^2\theta = \frac{1}{2^6}[sin\, 7\theta - 3sin\, 5\theta + \sin 3\theta + 5sin\theta]$$

Check your understanding!

- **Prove that** $\boldsymbol{sin^5\theta = \frac{1}{16}(sin5\theta - 5sin3\theta + 10sin\theta)}$

- **Write the expansion of cos^4 $\boldsymbol{\theta}$ sin^3 $\boldsymbol{\theta}$ in a series of sines of multiples of**

- Write the expansion of sin^4 $\boldsymbol{\theta}$cos^5 $\boldsymbol{\theta}$ in a series of cosines of multiples of $\boldsymbol{\theta}$

- **Prove that**

$$\boldsymbol{cos^9\theta = \frac{1}{256}(\cos 9\theta + 9\cos 7\theta + 36\cos 5\theta + 84\cos 3\theta + 126\,cos\theta)}$$