eLearning by Dr. M.G.R Educational and Research Institute

TOPIC : PARTIAL & TOTAL DIFFERENTIAL COEFFICIENT

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Let Z be a function in two or more variables, it can be differentiated with respect to each of the variable by assuming that it varies only with that variable and others treated as constants. These differential Co-efficients are Known as Partial differential Co-efficient. They are denoted by $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial u}{\partial t}$ e.tc

If u $= e^x \sin y$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

<u>Solution:</u>

$$\frac{\partial u}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = e^x \cos y$$

<u>Chain Rule For Partial Differentiation</u>

$$u = f(x, y) \text{ and } x = f(s), y = f(s)$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

- If $\phi = f(y-z, z-x, x-y)$, show that $\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} = 0$

Solution:

Given $\phi = f(y-z, z-x, x-y)$

Let $u = y - z$

$v = z - x$

$w = x - y$

By chain rule,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{\partial \phi}{\partial u}(0) + \frac{\partial \phi}{\partial v}(-1) + \frac{\partial \phi}{\partial w}(1)$$

$$= \frac{\partial \phi}{\partial w} - \frac{\partial \phi}{\partial v} \quad \text{-----------------------}(1)$$

Similarly

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u} - \frac{\partial \phi}{\partial w} \quad \text{-----------------------}(2)$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial v} - \frac{\partial \phi}{\partial u} \quad \text{-----------------------}(3)$$

Adding (1) ,(2) and (3), we get,

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} = 0$$

If $v = (y-z)(z-x)(x-y)$, prove that $\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}=0$

Solution:

$$v = (y-z)(z-x)(x-y)$$

$$= (yz - z^2 - xy + xz)(x-y)$$

$$= xyz - xz^2 - x^2y + x^2z - y^2z + yz^2 + xy^2 - xyz$$

$$= xy^2 + yz^2 + x^2z - xz^2 - x^2y - y^2z$$

$$\frac{\partial v}{\partial x} = y^2 + 2xz - 2xy - z^2 \quad \text{-------------}(1)$$

$$\frac{\partial v}{\partial y} = 2xy + z^2 - x^2 - 2yz \quad \text{--------------}(2)$$

$$\frac{\partial v}{\partial z} = 2yz + x^2 - 2xz - y^2 \quad \text{----------------}(3)$$

Adding (1), (2) and (3)

$$\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}$$

$$= y^2 + 2xz - 2xy - z^2 + 2xy$$

$$+ z^2 - x^2 - 2yz$$

$$+2yz + x^2 - 2xz - y^2$$

$$=0$$

If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

<u>Solution:</u>

Given $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$

Let $u = f(p, q, r)$, where $p = \frac{x}{y}, q = \frac{y}{z}, r = \frac{z}{x}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}\left(\frac{-z}{x^2}\right)$$

$$= \frac{1}{y}\frac{\partial u}{\partial p} - \frac{z}{x^2}\frac{\partial u}{\partial r} \qquad \text{-----------------}(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$= \frac{\partial u}{\partial p}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial q}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial r}\,(0)$$

$$= \frac{1}{z}\frac{\partial u}{\partial q} - \frac{x}{y^2}\frac{\partial u}{\partial p} \qquad ----------------(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\,\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\,\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\,\frac{\partial r}{\partial z}$$

$$= \frac{\partial u}{\partial p}\,(0) + \frac{\partial u}{\partial q}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial r}\left(\frac{1}{x}\right)$$

$$= \frac{1}{x}\frac{\partial u}{\partial r} - \frac{y}{z^2}\frac{\partial u}{\partial q} \qquad ----------------(3)$$

Therefore, from (1), (2) and (3), we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{x}{y}\frac{\partial u}{\partial p} - \frac{z}{x}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial q} - \frac{x}{y}\frac{\partial u}{\partial p} + \frac{z}{x}\frac{\partial u}{\partial r} - \frac{y}{z}\frac{\partial u}{\partial q}$$

$$= 0 = \text{RHS}$$

- If $\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{u}, \boldsymbol{v})$, where $\boldsymbol{u} = \boldsymbol{x}^2 - \boldsymbol{y}^2$ and $\boldsymbol{v} = 2\boldsymbol{x}\boldsymbol{y}$, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(x^2 + y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right].$$

Solution:

Given $z = f(u,v)$

where $u = x^2 - y^2, v = 2xy$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}(2x) + \frac{\partial z}{\partial v}(2y)$$

$$= 2x\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v} \qquad -------------(1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}(-2y) + \frac{\partial z}{\partial v}(2x) \quad = -2y\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v} \qquad -------------(2)$$

Therefore, From (1) and (2), Squaring and addding, we get,

$$\text{LHS} = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(2x\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v}\right)^2 + \left(-2y\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}\right)^2$$

$$= 4x^2\left(\frac{\partial z}{\partial u}\right)^2 + 8xy\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + 4y^2\left(\frac{\partial z}{\partial v}\right)^2 + 4y^2\left(\frac{\partial z}{\partial u}\right)^2$$
$$- 8xy\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + 4x^2\left(\frac{\partial z}{\partial v}\right)^2$$

$$= \left(\frac{\partial z}{\partial u}\right)^2 [4x^2 + 4y^2] + \left(\frac{\partial z}{\partial v}\right)^2 [4x^2 + 4y^2]$$

$$= [4x^2 + 4y^2]\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4(x^2 + y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] = \text{RHS}$$

Let Z be a function in two variables x and y.If Z is continuous, then the total differential coefficient of Z is given by $dz = \frac{\partial Z}{\partial x}\, dx + \frac{\partial Z}{\partial y}\, dy$

Find the total differential coefficient of the function $u = tan\,(3x - y + 2z)$

Solution:

Given, $u = tan\,(3x - y + 2z)$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \qquad \text{-------------------(1)}$$

$$\frac{\partial u}{\partial x} = 3sec^2(3x - y + 2z)$$

$$\frac{\partial u}{\partial y} = -sec^2(3x - y + 2z)$$

$$\frac{\partial u}{\partial z} = 2sec^2(3x - y + 2z)$$

Substituting in (1)

$$du = 3sec^2(3x - y + 2z)dx - sec^2(3x - y + 2z)dy + 2sec^2(3x - y + 2z)dz$$

$$du = sec^2(3x - y + 2z)(3dx - dy + 2dz)$$

Find $\frac{du}{dt}$, if $u = x^3y^2 + x^2y^3$ *where* $x = at^2, y = 2at$

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= (3x^2y^2 + 2xy^3)(2at) + (2x^3y + 3x^2y^2)(2a)$$

$$= (3a^2t^4 4a^2t^2 + 2at^2 8a^3t^3)(2at) + (2a^3t^6 2at + 3a^2t^4 4a^2t^2)(2a)$$

$$= a^4t^5(3t + 4)(2at) + 4a^4t^6(t + 3)(2a)$$

$$= 8a^5t^6(3t + 4) + 8a^5t^6(t + 3)$$

$$= 8a^5t^6(3t + 4 + t + 3)$$

$$= 8a^5t^6(4t + 7)$$

If $\boldsymbol{u} = \boldsymbol{sin}^{-1}(\boldsymbol{x} - \boldsymbol{y})$, where $\boldsymbol{x} = \boldsymbol{3t}$ and $\boldsymbol{y} = \boldsymbol{4t}^3$. Show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$

Solution:

Given, $u = sin^{-1}(x - y)$

where $x = 3t$ *and* $y = 4t^3$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}(3) - \frac{1}{\sqrt{1-(x-y)^2}}(12t^2) = \frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

Now $1 - (x - y)^2 = 1 - (3t - 4t^3)^2$

$$= 1 - t^2(3 - 4t^2)^2$$

$$= 1 - t^2(9 - 24t^2 + 16t^4)$$

$$= 1 - 9t^2 + 24t^4 - 16t^6$$

$$= 1 - t^2 - 8t^2 + 8t^4 + 16t^4 - 16t^6$$

$$= (1 - t^2)(1 - 8t^2 + 16t^4)$$

$$= (1 - t^2)(1 - 4t^2)^2$$

$$\frac{du}{dt} = \frac{3(1 - 4t^2)}{\sqrt{(1 - t^2)(1 - 4t^2)^2}} = \frac{3}{\sqrt{1 - t^2}}$$

Check your understanding!

- If $u = f(x, y)$, where $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

- If $u = f(x^2+2yz, y^2+2zx)$, prove that

$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$$

- Find $\frac{du}{dt}$, if $u = \log(x + y + z)$, where $x = e^{-t}, y = \sin t, z = \cos t$

- Find $\frac{du}{dt}$, if $u = e^{xy}$, where $x = (a^2 - t^2)^{1/2}$, $y = \sin^3 t$

Find $\frac{du}{dt}$, if $u = \frac{x}{y}$, where $x = e^t$, and $y = \log t$